Name of college - S.S. College, J-Bad

Deptt - Mathematics

Topic - Continuity and Differentiability (Real Analysis) (Rolle's theorem)

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Time - 9:00 A.M 10:30 A.M

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B.Sc II (HONS + Sub)

Rolle's theorem

Rolle's theorem :→

Statement :→ If $f(x)$ be a real valued function such that

(i) $f(x)$ is continuous in the closed interval $[a,b]$

(ii) $f(x)$ is differentiable over $]a,b[$

(iii) $f(a) = f(b)$.

then there exists at least a point c at which $f'(c) = 0$ where $c \in (a,b)$

Proof :→ case I When $f(x)$ is constant function

Let $f(x) = K$

then conditions (i) (ii) and (iii) are satisfied

Also $f'(c) = 0 \ \forall \ x \in [a,b)$

thus the theorem is valid in this case.

Case II :→ If $f(x)$ is variable function.

Since $f(x)$ is continuous over a closed interval $[a,b]$ $f(x)$ will attain its (absolute). Maximum as well as Minimum over $[a,b]$

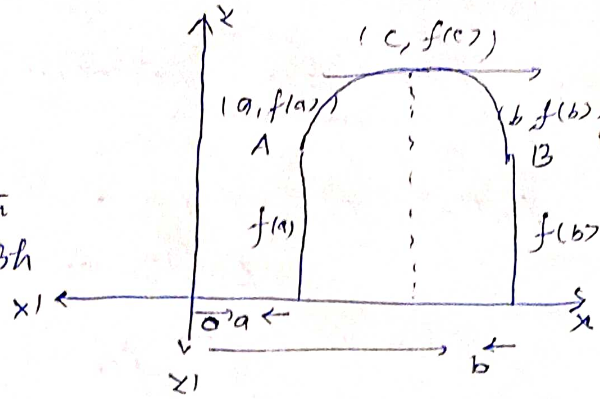
But since $f(x)$ is not constant function and $f(a) = f(b)$,

either the maximum or minimum will be attained at a point $c \in [a,b]$. Now since there is an extremum at c which is not an end point of the interval $[a,b]$,

we have $f'(c) = 0$

Geometrical significance of Rolle's theorem —

Rolle's theorem says that if a function has a continuous graph on the interval $[a,b]$ and the graph has a Tangent at every point of (a,b), then the graph of the function must have a Tangent parallel to the x-axis at least at on a point between a and b as shown in the graph



NOTE — The converse of Rolle's theorem is not True

Algebraic interpretation of Rolle's theorem :-

If $f(x)$ be a polynomial in x and $x=a$, $x=b$ be the two roots of the Equation $f(x)=0$, then we From Rolle's theorem we Find that at least one root of the Equation $f'(x)=0$ lies between a and b.

Problem ! Discuss the application of Rolle's theorem to the function $f(x)=x^{2/3}$ in $(-1,1)$

Solution — Since $f(x) = x^{2/3}$

$\Rightarrow f'(x) = \frac{2}{3}x^{-1/3}$ (By the rule of ordinary differentiation)

$$\therefore \lim_{x\to 0} f'(x) = \lim_{h\to 0} \frac{2}{3}(0+h)^{-1/3} = +\infty$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{2/3}-0}{h}$$

$$= \lim_{h\to 0} \frac{1}{h^{1/3}} = \infty$$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{(-h)^{2/3}}{(-h)} = -\infty$$

$\because Rf'(0) \neq Lf'(0)$

$\therefore f'(0)$ does not Exist in the open.

open interval

$(-1, 1)$

thus the Rolle's theorem is not applicable although $f(-1) = f(1) = 1$ and $f(x)$ is continuous in the closed interval $(-1, 1]$

2. Discuss the applicability of Rolle's theorem to the function $f(x) = x^2$ in $(-1, 1)$

Sol^n :→ Since $f(x) = x^2$

$\Rightarrow f(-1) = 1$ and $f(1) = 1$

$$Rf'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x$$

$\therefore Rf'(x) = 2x$

Also $$Lf'(x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h\to 0} \frac{(x-h)^2 - x^2}{-h}$$

$$= \lim_{h\to 0} \frac{(x-h+x)(x-h-x)}{-h}$$

$$= \lim_{h\to 0} \frac{(2x-h)(-h)}{-h} = 2x$$

$\therefore$ Here $Lf'(x) = Rf'(x)$

$\therefore$ $f'(x)$ exists $\forall\, x \in (-1,1)$

Also $f(x)$ is continuous in $[-1,1]$

[As $f(x)$ is differentiable for all values of x in $(-1,1)$]

Thus all the conditions of Rolle's theorem are satisfied

$\therefore f'(x) = 0$ for at least one value of x where $-1 \le x \le 1$

* Discuss applicability of Rolle's theorem to the function $f(x) = |x|$ in $[-1,1]$

Solution $\rightarrow$ Here $f(x) = |x|$

$\Rightarrow f(-1) = |-1| = 1$ Also $f(1) = |1| = 1$

Now $f(x) = |x|$

$f(0+h) = |0+h| = h.$ $\quad f(0-h) = |0-h| = h$

$f(0) = |0| = 0$

$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} h = 0$

$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} h = 0.$

Also $f(0) = 0$

therefore $f(x)$ is continuous at $x=0$ and therefore, $f(x)$ is continuous at every point of $[-1,1]$

Also $Rf'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{h-0}{h} = 1$

$Lf'(0) = \lim_{h\to 0} \dfrac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \dfrac{h}{-h} = -1$

Here $Rf'(0) \neq Lf'(0) \Rightarrow f(x)$ is not differentiable in $(-1,1)$ and therefore Rolle's theorem is not applicable.